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О СВЕРХРАЗРЕШИМОМ КОРАДИКАЛЕ ВЗАИМНО ПЕРЕСТАНОВОЧНЫХ ПОДГРУПП

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ON THE SUPERSOLUBLE RESIDUAL OF MUTUALLY PERMUTABLE PRODUCTS

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Доказывается, что если группа $G = AB$ является произведением взаимно перестановочных сверхразрешимых подгрупп A и B, то сверхразрешимый корадикал группы G совпадает с нильпотентным корадикалом коммутанта G'.

Ключевые слова: *конечная группа*, *сверхрарешимая подгруппа*, *взаимно перестановочные подгруппы*, *корадикал*.

We prove that if a group $G = AB$ is the mutually permutable product of the supersoluble subgroups A and B, then the supersoluble residual of G coincides with the nilpotent residual of the derived subgroup G'.

Keywords: *finite group*, *supersoluble subgroup*, *mutually permutable product*, *residual*.

1 Preliminaries

All groups in this paper are finite. Formations of all abelian, nilpotent and supersoluble groups is denoted by $\mathfrak{A}$, $\mathfrak{N}$ and $\mathfrak{U}$ respectively. If $\mathfrak{F}$ is a formation and G is a group, then $G^{\mathfrak{F}}$ is the $\mathfrak{F}$-residual of G, i. e., the smallest normal subgroup of G with quotient in $\mathfrak{F}$. If $\mathfrak{X}$ and $\mathfrak{F}$ are hereditary formations, then, according to [1, p. 337–338], the product

$$\mathfrak{X}\mathfrak{F} = \{G \in \mathfrak{E} \mid G^{\mathfrak{F}} \in \mathfrak{X}\}$$

is also a hereditary formation. A Fitting class which is also a formation is called a Fitting formation.

We need the following lemmas.

***Lemma* 1.1** [2, 4.8]. *Let $G = AB$ be the product of two subgroups A and B. Then*

(1) $[A,B] = \langle [a,b] \mid a \in A, b \in B \rangle \lhd G$;

(2) *if* $A_1 \lhd A$, *then* $A_1[A,B] \lhd G$;

(3) $G' = A'B'[A,B]$.

***Lemma* 1.2** [1, IV.11.7]. *Let $\mathfrak{F}$ and $\mathfrak{H}$ be formations, G be a group and $K \lhd G$. Then*

(1) $(G/K)^{\mathfrak{F}} = G^{\mathfrak{F}}K/K$;

(2) $G^{\mathfrak{F}\mathfrak{H}} = (G^{\mathfrak{H}})^{\mathfrak{F}}$;

(3) *if* $\mathfrak{H} \subseteq \mathfrak{F}$, *then* $G^{\mathfrak{F}} \subseteq G^{\mathfrak{H}}$.

If H is a subgroup of a group G, then H^G denotes the smallest normal subgroup of G containing H.

***Lemma* 1.3** [2, 5.31]. *Let H be a subnormal subgroup of a group G. If H belongs to a Fitting class $\mathfrak{F}$, then $H^G \in \mathfrak{F}$. In particular,*

(1) *if H is nilpotent, then H^G is also nilpotent*;

(2) *if H is p-nilpotent, then H^G is also p-nilpotent.*

***Lemma* 1.4.** *Let $G = AB$ be the product of the supersoluble subgroups A and B. Then $G^{\mathfrak{U}} \le [A,B]$.*

Proof. By Lemma 1.1 (1,3) and Lemma 1.2 (1),

$$(G/[A,B])' = G'[A,B]/[A,B] =$$
$$= A'B'[A,B]/[A,B] =$$
$$= (A'[A,B]/[A,B])(B'[A,B]/[A,B]).$$

The subgroups

$$(A'[A,B])/[A,B] \simeq A'/(A' \cap [A,B]),$$
$$(B'[A,B])/[A,B] \simeq B'/(B' \cap [A,B])$$

are nilpotent [3, VI.9.1] and normal in $G/[A,B]$ by Lemma 1.1 (3), so $(G/[A,B])'$ is nilpotent. By Lemma 1.1 (3), $A[A,B]$ and $B[A,B]$ are normal in G. In view of the Baer Theorem [4], $G/[A,B]$ is supersoluble. Hence, $G^{\mathfrak{U}} \le [A,B]$. □

***Lemma* 1.5** [1, II.2.12]. *Let $\mathfrak{X}$ be a Fitting formation, and let $G = AB$ be the product of normal subgroups A and B. Then $G^{\mathfrak{X}} = A^{\mathfrak{X}}B^{\mathfrak{X}}$.*

2 On the $\mathfrak{U}$-residual of mutually permutable product

A group $G = AB$ is called the mutually permutable product of subgroups A and B if $UB = BU$ and $AV = VA$ for all $U \le A$ and $V \le B$. Such groups were studied in [5]–[8], see also [9].

We prove the following theorem.

***Theorem* 2.1.** *Let $G = AB$ be the mutually permutable product of the supersoluble subgroups A and B. Then $G^{\mathfrak{U}} = (G')^{\mathfrak{N}} = [A,B]^{\mathfrak{N}}$.*

Proof. By Lemma 1.4, $G^{\mathfrak{U}} \le [A,B]$. Since $\mathfrak{U} \subseteq \mathfrak{N}\mathfrak{A}$ [3, VI.9.1], by Lemma 1.2 (2,3), we have

$$G^{(\mathfrak{N}\mathfrak{A})} = (G^{\mathfrak{A}})^{\mathfrak{N}} = (G')^{\mathfrak{N}} \le G^{\mathfrak{U}}.$$

Verify the reverse inclusion. Since

$$(G/(G')^{\mathfrak{N}})' = G'(G')^{\mathfrak{N}}/(G')^{\mathfrak{N}} = G'/(G')^{\mathfrak{N}}$$

is nilpotent,

$$G/(G')^{\mathfrak{N}} = A(G')^{\mathfrak{N}}/(G')^{\mathfrak{N}} \cdot B(G')^{\mathfrak{N}}/(G')^{\mathfrak{N}}$$

is supersoluble in view of [5, Theorem 3.8] and $G^{\mathfrak{U}} \le (G')^{\mathfrak{N}}$. Thus, $G^{\mathfrak{U}} = (G')^{\mathfrak{N}}$.

By Lemma 1.1 (3),

$$G' = A'B'[A,B] = (A')^G (B')^G [A,B].$$

The subgroups A' and B' are subnormal in G by [8, Theorem 1] and nilpotent, therefore $(A')^G(B')^G$ is normal in G and nilpotent by Lemma 1.3 (1). In view of Lemma 1.5 with $\mathfrak{X} = \mathfrak{N}$, we get

$$G^{\mathfrak{U}} = (G')^{\mathfrak{N}} = ((A')^G(B')^G)^{\mathfrak{N}}[A,B]^{\mathfrak{N}} = [A,B]^{\mathfrak{N}}. \quad \square$$

***Corollary* 2.1.1** *Let $G = AB$ be the mutually permutable product of the supersoluble subgroups A and B. If $[A,B]$ is nilpotent, then G is supersoluble.*

The class of all *p*-nilpotent groups coincides with the product $\mathfrak{E}_{p'}\mathfrak{N}_p$, where $\mathfrak{N}_p$ is the class of all *p*-groups and $\mathfrak{E}_{p'}$ is the class of all p'-groups. A group G is *p*-supersoluble if all chief factors of G having order divisible by the prime p are exactly of order p. The derived subgroup of a *p*-supersoluble group is *p*-nilpotent [3, VI.9.1. (a)]. The class of all *p*-supersoluble groups is denoted by $p\mathfrak{U}$. It's clear that $\mathfrak{E}_{p'}\mathfrak{N}_p \subseteq p\mathfrak{U} \subseteq \mathfrak{E}_{p'}\mathfrak{N}_p\mathfrak{A}$.

***Theorem* 2.2.** *Let $G = AB$ be the mutually permutable product of the p-supersoluble subgroups A and B. Then $G^{p\mathfrak{U}} = (G')^{\mathfrak{E}_{p'}\mathfrak{N}_p} = [A,B]^{\mathfrak{E}_{p'}\mathfrak{N}_p}$.*

Proof. By Lemma 1.2,

$$(G')^{\mathfrak{E}_{p'}\mathfrak{N}_p} = (G^{\mathfrak{A}})^{\mathfrak{E}_{p'}\mathfrak{N}_p} = G^{\mathfrak{E}_{p'}\mathfrak{N}_p\mathfrak{A}} \le G^{p\mathfrak{U}}.$$

Verify the reverse inclusion. The quotient group

$$G/(G')^{\mathfrak{E}_{p'}\mathfrak{N}_p} =$$
$$= (A(G')^{\mathfrak{E}_{p'}\mathfrak{N}_p}/(G')^{\mathfrak{E}_{p'}\mathfrak{N}_p})(B(G')^{\mathfrak{E}_{p'}\mathfrak{N}_p}/(G')^{\mathfrak{E}_{p'}\mathfrak{N}_p})$$

is the mutually permutable product of the *p*-supersoluble subgroups $A(G')^{\mathfrak{E}_{p'}\mathfrak{N}_p}/(G')^{\mathfrak{E}_{p'}\mathfrak{N}_p}$ and $B(G')^{\mathfrak{E}_{p'}\mathfrak{N}_p}/(G')^{\mathfrak{E}_{p'}\mathfrak{N}_p}$. The derived subgroup

$$(G/(G')^{\mathfrak{E}_{p'}\mathfrak{N}_p})' = G'(G')^{\mathfrak{E}_{p'}\mathfrak{N}_p}/(G')^{\mathfrak{E}_{p'}\mathfrak{N}_p} =$$
$$= G'/(G')^{\mathfrak{E}_{p'}\mathfrak{N}_p}$$

is *p*-nilpotent. By [8, Corollary 5], $G/(G')^{\mathfrak{E}_{p'}\mathfrak{N}_p}$ is *p*-supersoluble, consequently, $G^{p\mathfrak{U}} \le (G')^{\mathfrak{E}_{p'}\mathfrak{N}_p}$. Thus, $G^{p\mathfrak{U}} = (G')^{\mathfrak{E}_{p'}\mathfrak{N}_p}$.

By Lemma 1.1 (3),

$$G' = A'B'[A,B] = (A')^G (B')^G [A,B].$$

The subgroups A' and B' are subnormal in group G [8, Theorem 1] and *p*-nilpotent [3, VI.9.1 (a)], hence $(A')^G(B')^G$ normal in G and *p*-nilpotent by Lemma 1.3 (2). In view of Lemma 1.5 with $\mathfrak{X} = \mathfrak{E}_{p'}\mathfrak{N}_p$, we get

$$G^{p\mathfrak{U}} = (G')^{\mathfrak{E}_{p'}\mathfrak{N}_p} =$$
$$= ((A')^G(B')^G)^{\mathfrak{E}_{p'}\mathfrak{N}_p})[A,B]^{\mathfrak{E}_{p'}\mathfrak{N}_p} = [A,B]^{\mathfrak{E}_{p'}\mathfrak{N}_p}. \quad \square$$

***Corollary* 2.2.1.** *Let $G = AB$ be the mutually permutable product of the p-supersoluble subgroups A and B. If $[A,B]$ is p-nilpotent, then G is p-supersoluble.*

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